



Double integrals involving product of two I -functions

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Abstract: In this paper two double integrals involving product of two I -Functions of one variable are established by using Mellin Transform. Special cases include the results of Prabha Singh[4].

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1 Results used

The following well known results are used in the proofs of the main results without explicit reference. Here and also elsewhere in the paper the symbol $\Re z$ shall denote the real part of the complex number z .

1. For $\Re \gamma > 0$, $\Re \mu < -\Re \gamma + 1$ we have (Erdelyi [1, p.350]),

$$\int_1^{\infty} x^{\mu-1}(x-1)^{\gamma-1} dx = \frac{\Gamma(\gamma)\Gamma(1-\gamma-\mu)}{\Gamma(1-\mu)}. \quad (1)$$

2. For $0 < \Re \mu < \Re \lambda$ we have (Erdelyi [2, pp.201-233]),

$$\int_y^{\infty} x^{-\lambda}(x-y)^{\mu-1} dx = \frac{\Gamma(\mu)\Gamma(\lambda-\mu)}{\Gamma(\lambda)} y^{\mu-\lambda}. \quad (2)$$

3. For $\Re \gamma > 0$, $\Re \rho > \Re \gamma$ we have,

$$\int_0^{\infty} x^{\gamma-1}(x+y)^{-\rho} dx = \frac{\Gamma(\gamma)\Gamma(\rho-\gamma)}{\Gamma(\rho)} y^{\gamma-\rho}. \quad (3)$$

4. For $0 < \Re \mu < \Re \lambda$, $|\arg \frac{y}{a}| < \pi$ we have,

$$\int_y^{\infty} (x+a)^{-\lambda}(x-y)^{\mu-1} dx = \frac{\Gamma(\mu)\Gamma(\lambda-\mu)}{\Gamma(\lambda)} (y+a)^{\mu-\lambda}. \quad (4)$$

2 Introduction

The I -function of one variable is defined by Saxena [5] in the following manner:

$$\begin{aligned}
 I(z) &= I_{p_i, q_i; r}^{m, n} \left[z \left| \begin{array}{l} 1(a_j, \alpha_j)_{n; n+1} (a_{ji}, \alpha_{ji})_{p_i} \\ 1(b_j, \beta_j)_{m; m+1} (b_{ji}, \beta_{ji})_{q_i} \end{array} \right. \right] \\
 &= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\sum_{i=1}^r \left[\prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} \xi) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} \xi) \right]} z^\xi d\xi, \quad (5)
 \end{aligned}$$

where the parameters defining the function satisfy the following conditions:

- $p_i (i = 1, 2, \dots, r)$, $q_i (i = 1, 2, \dots, r)$, m , n are integers satisfying

$$0 \leq n \leq p_i, \quad 0 \leq m \leq q_i;$$

- r is finite;
- $\alpha_j, \beta_j, \alpha_{ji}$ and β_{ji} are real and positive;
- a_j, b_j, a_{ji} and b_{ji} are complex numbers such that

$$a_j(b_h + v) \neq \beta_h(a_j - 1 - k),$$

for $v, k = 0, 1, 2, \dots, h = 1, 2, \dots, m, j = 1, 2, \dots, n$;

- L is a contour running from $-i\infty$ to $+i\infty$ in the complex ξ -plane such that the points

$$\xi = \frac{(a_j - 1 - k)}{\alpha_j}, \quad j = 1, 2, \dots, n, \quad v = 0, 1, 2, \dots$$

$$\xi = \frac{(b_j + v)}{\beta_j}, \quad j = 1, 2, \dots, m, \quad v = 0, 1, 2, \dots$$

lie to the left hand and right hand sides of L .

Integral (5) converges absolutely in the ξ -plane, if

- $A > 0, |\arg z| < \frac{\pi A}{2}$
- $A \geq 0, |\arg z| \geq \frac{\pi A}{2}, \Re(B + 1) < 0$

where A and B are defined as follows:

$$A = \sum_{j=1}^n \alpha_j + \sum_{j=1}^m \beta_j - \max_{1 \leq i \leq r} \left[\sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=m+1}^{q_i} \beta_{ji} \right], \quad (6)$$

$$B = \sum_{j=1}^m b_j - \sum_{j=1}^n a_j - \min_{1 \leq i \leq r} \left[\sum_{j=n+1}^{p_i} a_{ji} - \sum_{j=m+1}^{q_i} b_{ji} + \frac{p_i}{2} - \frac{q_i}{2} \right]. \quad (7)$$

3 Main results

In the next two subsections we present the main results of this paper which are two integral formulas involving the product of two I -functions. Two special cases of each of these integral formulas are also presented. One of these special cases is a result proved by Prabha Singh [4].

3.1 Integral formula 1

The following formula is one of the main results of this paper.

$$\begin{aligned} & \int_y^\infty \int_0^\infty x^{-\lambda} (x-y)^{\mu-1} z^{\eta-1} I_{p_i, q_i; r}^{m, n} \times \\ & \quad \left[ux^\rho (x-y)^\gamma z^{-\sigma} \middle| \begin{matrix} I_1 \\ I_2 \end{matrix} \right] I_{P_i, Q_i; r}^{M, N} \left[bz^\delta \middle| \begin{matrix} I_3 \\ I_4 \end{matrix} \right] dx dz \\ & = \frac{b^{-\frac{\eta}{\delta}} y^{(\mu-\lambda)}}{\delta} I_{p_i+P_i+2, q_i+Q_i+1; r}^{m+M+1, n+N+1} \left[ub^{\frac{\sigma}{\delta}} y^{(\gamma+\rho)} \middle| \begin{matrix} I_5 \\ I_6 \end{matrix} \right], \end{aligned} \quad (8)$$

where

$$\begin{aligned} I_1 &= {}_1(a_j, \alpha_j)_n ; {}_{n+1}(a_{ji}, \alpha_{ji})_{p_i} \\ I_2 &= {}_1(b_j, \beta_j)_m ; {}_{m+1}(b_{ji}, \beta_{ji})_{q_i} \\ I_3 &= {}_1(c_j, \gamma_j)_N ; {}_{N+1}(c_{ji}, \gamma_{ji})_{P_i} \\ I_4 &= {}_1(d_j, \delta_j)_M ; {}_{M+1}(d_{ji}, \delta_{ji})_{Q_i} \\ I_5 &= (1-\mu, \gamma), {}_1(a_j, \alpha_j)_n, {}_1\left(c_j + \frac{\eta\gamma_j}{\delta}, \frac{\sigma\gamma_j}{\delta}\right)_N ; \\ & \quad {}_{N+1}\left(c_{ji} + \frac{\eta\gamma_{ji}}{\delta}, \frac{\sigma\gamma_{ji}}{\delta}\right)_{P_i}, {}_{n+1}(a_{ji}, \alpha_{ji})_{p_i}, (\lambda, \rho) \\ I_6 &= (\lambda-\mu, \rho+\gamma), {}_1(b_j, \beta_j)_m, {}_1\left(d_j + \frac{\eta\delta_j}{\delta}, \frac{\sigma\delta_j}{\delta}\right)_M ; \\ & \quad {}_{M+1}\left(d_{ji} + \frac{\eta\delta_{ji}}{\delta}, \frac{\sigma\delta_{ji}}{\delta}\right)_{Q_i}, {}_{m+1}(b_{ji}, \beta_{ji})_{q_i}. \end{aligned}$$

The following are conditions of convergence:

- i. $\sigma > 0, \rho > 0, \eta > 0, \gamma > 0, \delta > 0, \mu > 0, \lambda > 0, i = 1, 2, \dots, r.$
- ii. $0 < \Re(\mu) < \Re(\lambda).$
- iii. $\Re\left(\lambda - \mu - \frac{(\gamma+\rho)(a_j-1)}{\alpha_i}\right) > 0$ for $j = 1, 2, \dots, n.$
- iv. $\Re\left(\eta + \frac{\delta(c_i-1)}{\gamma_i} - \frac{\sigma b_j}{\beta_j}\right) < 0$ for $i = 1, 2, \dots, N, j = 1, 2, \dots, m.$

- v. $\Re \left(\eta + \frac{\delta(d_j)}{\delta_j} + \frac{\sigma(1-a_i)}{\alpha_i} \right) > 0$ for $i = 1, 2, \dots, n, j = 1, 2, \dots, M$.
- vi. $\Re \left(\mu + \frac{\gamma b_j}{\beta_j} \right) > 0$, for $j = 1, 2, \dots, m$.
- vii. Let A and B be given by Eq.(6) and Eq.(7) respectively.
 - (a) $A > 0, |\arg u| < \frac{\pi A}{2}$;
 - (b) $A \geq 0, |\arg u| \geq \frac{\pi A}{2}, |\Re(B + 1)| < 0$.
- viii. Define A' and B' by

$$A' = \sum_{j=1}^N \gamma_j + \sum_{j=1}^M \delta_j - \max_{1 \leq i \leq r} \left[\sum_{j=N+1}^{P_i} \gamma_{ji} + \sum_{j=M+1}^{Q_i} \delta_{ji} \right] \quad (9)$$

$$B' = \sum_{j=1}^M d_j - \sum_{j=1}^N c_j - \min_{1 \leq i \leq r} \left[\sum_{j=N+1}^{P_i} c_{ji} - \sum_{j=M+1}^{Q_i} d_{ji} + \frac{P_i}{2} - \frac{Q_i}{2} \right]. \quad (10)$$

We must have

- (a) $A' > 0, |\arg b| < \frac{\pi A'}{2}$;
- (b) $A' \geq 0, |\arg b| \geq \frac{\pi A'}{2}, \Re(B' + 1) < 0$.

Proof

Let

$$\Delta = \int_y^\infty \int_0^\infty x^{-\lambda} (x-y)^{\mu-1} z^{\eta-1} I_{P_i, Q_i; r}^{m, n} \left[\begin{matrix} I_1 \\ I_2 \end{matrix} \middle| \begin{matrix} I_3 \\ I_4 \end{matrix} \right] dx dz,$$

where I_1, I_2, I_3 and I_4 are same as in (8). Express the I -function as a contour integral using (5) and interchange the order of integrations. Then Δ becomes

$$\frac{1}{2\pi i} \int_L u^\xi \left\{ \int_y^\infty x^{\rho\xi-\lambda} (x-y)^{\gamma\xi+\mu-1} dx \right\} \left\{ \int_0^\infty z^{\eta-\sigma\xi-1} I_{P_i, Q_i; r}^{M, N} \left[\begin{matrix} I_3 \\ I_4 \end{matrix} \middle| bz^\delta \right] dz \right\} \theta(\xi) d\xi, \quad (11)$$

where $\theta(\xi)$ is same as in Eq.(8). Now being used Eq.(2) in Eq.(11) it can be reduced to:

$$\frac{b^{-\frac{\eta}{\delta}}}{\delta} y^{\mu-\lambda} \frac{1}{2\pi i} \int_L b^{\frac{\sigma\xi}{\delta}} y^{(\gamma+\rho)\xi} u^\xi \theta(\xi) \frac{\{\Gamma(\mu + \gamma\xi)\Gamma(\lambda - \mu - (\rho + \gamma)\xi)\}}{\Gamma(\lambda - \rho\xi)} \times$$

$$\frac{\prod_{j=1}^M \Gamma\left(d_j + \frac{\eta\delta_j}{\delta} - \frac{\sigma\delta_j\xi}{\delta}\right) \prod_{j=1}^N \Gamma\left(1 - c_j - \frac{\eta\gamma_j}{\delta} + \frac{\sigma\gamma_j\xi}{\delta}\right)}{\sum_{i=1}^r \left[\prod_{j=M+1}^{Q_i} \Gamma\left(1 - d_{ji} - \frac{\eta\delta_{ji}}{\delta} + \frac{\sigma\delta_{ji}\xi}{\delta}\right) \prod_{j=N+1}^{P_i} \Gamma\left(c_{ji} + \frac{\eta\gamma_{ji}}{\delta} - \frac{\sigma\gamma_{ji}\xi}{\delta}\right) \right]} d\xi. \quad (12)$$

Applying (5) in (12), the right hand side of (8) is obtained.

3.1.1 Special Cases

1. The case $r = 1$

Eq.(8) reduces to:

$$\begin{aligned} & \int_y^\infty \int_0^\infty x^{-\lambda} (x-y)^{\mu-1} z^{\eta-1} H_{p,q}^{m,n} \\ & \left[ux^\rho (x-y)^\gamma z^{-\sigma} \left| \begin{matrix} I_1 \\ I_2 \end{matrix} \right. \right] H_{P,Q}^{M,N} \left[bz^\delta \left| \begin{matrix} I_3 \\ I_4 \end{matrix} \right. \right] dx dz \\ & = \frac{b^{-\frac{\eta}{\delta}} y^{(\mu-\lambda)}}{\delta} H_{p+P+2, q+Q+1}^{m+M+1, n+N+1} \left[ub^{\frac{\sigma}{\delta}} y^{(\gamma+\rho)} \left| \begin{matrix} I_5 \\ I_6 \end{matrix} \right. \right], \end{aligned} \quad (13)$$

where

$$\begin{aligned} I_1 &= {}_1(a_j, \alpha_j)_p, \\ I_2 &= {}_1(b_j, \beta_j)_q, \\ I_3 &= {}_1(c_j, \gamma_j)_P, \\ I_4 &= {}_1(d_j, \delta_j)_Q, \\ I_5 &= (1 - \mu, \gamma)_{,1} (a_j, \alpha_j)_n, {}_1\left(c_j + \frac{\eta\gamma_j}{\delta}, \frac{\sigma\gamma_j}{\delta}\right)_P, {}_{n+1}(a_j, \alpha_j)_p, (\lambda, \rho), \\ I_6 &= (\lambda - \mu, \rho + \gamma)_{,1} (b_j, \beta_j)_m, {}_1\left(d_j + \frac{\eta\delta_j}{\delta}, \frac{\sigma\delta_j}{\delta}\right)_Q, {}_{m+1}(b_j, \beta_j)_q. \end{aligned}$$

The conditions of convergence are the following.

- i. $\delta > 0, \mu > 0, \lambda > 0, \gamma > 0, \sigma > 0, \rho > 0, \eta > 0$.
- ii. $0 < \Re(\mu) < \Re(\lambda)$,
- iii. $\Re\left(\lambda - \mu - \frac{(\gamma+\rho)(a_j-1)}{\alpha_i}\right) > 0$, for $j = 1, 2, \dots$
- iv. $\Re\left(\eta + \frac{\delta(c_i-1)}{\gamma_i} - \frac{\sigma b_j}{\beta_j}\right) < 0$, for $i = 1, 2, \dots, N, j = 1, 2, \dots, m$;
- v. $\Re\left(\eta + \frac{\delta(d_j)}{\delta_j} + \frac{\sigma(1-a_i)}{\alpha_i}\right) > 0$, for $i = 1, 2, \dots, n, j = 1, 2, \dots, M$.
- vi. $\Re\left(\mu + \frac{\gamma b_j}{\beta_j}\right) > 0$, for $j = 1, 2, \dots, m$.

2. **The case $\delta = 1$**

Eq.(13) reduces to the following.

$$\int_y^\infty \int_0^\infty x^{-\lambda} (x-y)^{\mu-1} z^{\eta-1} H_{p,q}^{m,n} \left[\begin{matrix} ux^\rho (x-y)^\gamma z^{-\sigma} \\ I_1 \\ I_2 \end{matrix} \middle| \begin{matrix} I_1 \\ I_2 \end{matrix} \right] H_{P,Q}^{M,N} \left[\begin{matrix} bz \\ I_3 \\ I_4 \end{matrix} \middle| \begin{matrix} I_3 \\ I_4 \end{matrix} \right] dx dz$$

$$= b^{-\eta} y^{(\mu-\lambda)} H_{p+P+2,q+Q+1}^{m+M+1,n+N+1} \left[\begin{matrix} ub^\sigma y^{(\gamma+\rho)} \\ I_5 \\ I_6 \end{matrix} \middle| \begin{matrix} I_5 \\ I_6 \end{matrix} \right], \quad (14)$$

where

$$I_1 = {}_1(a_j, \alpha_j)_p,$$

$$I_2 = {}_1(b_j, \beta_j)_q,$$

$$I_3 = {}_1(c_j, \gamma_j)_P,$$

$$I_4 = {}_1(d_j, \delta_j)_Q,$$

$$I_5 = (1 - \mu, \gamma), {}_1(a_j, \alpha_j)_n, {}_1(c_j + \eta\gamma_j, \sigma\gamma_j)_P, {}_{n+1}(a_j, \alpha_j)_p, (\lambda, \rho),$$

$$I_6 = (\lambda - \mu, \rho + \gamma), {}_1(b_j, \beta_j)_m, {}_1(d_j + \eta\delta_j, \sigma\delta_j)_Q, {}_{m+1}(b_j, \beta_j)_q.$$

provided the conditions are similar to that of (13) with $\delta = 1$. (14) agrees with the result given by Prabha Singh [4].

3.2 Integral Formula 2

The following formula is second main results of this paper.

$$\int_y^\infty \int_0^\infty (x+a)^{-\lambda} (x-y)^{\mu-1} z^{\eta-1} I_{p_i, q_i; r}^{m, n} \left[\begin{matrix} u(x+a)^\rho (x-y)^\gamma z^\sigma \\ I_1 \\ I_2 \end{matrix} \middle| \begin{matrix} I_1 \\ I_2 \end{matrix} \right] I_{P_i, Q_i; r}^{M, N} \left[\begin{matrix} bz^\delta \\ I_3 \\ I_4 \end{matrix} \middle| \begin{matrix} I_3 \\ I_4 \end{matrix} \right] dx dz$$

$$= \frac{b^{-\frac{\eta}{\delta}} (y+a)^{(\mu-\lambda)}}{\delta} I_{p_i+Q_i+2, q_i+P_i+1; r}^{m+N+1, n+M+1} \left[\begin{matrix} ub^{-\frac{\sigma}{\delta}} (y+a)^{(\gamma+\rho)} \\ I_5 \\ I_6 \end{matrix} \middle| \begin{matrix} I_5 \\ I_6 \end{matrix} \right], \quad (15)$$

where

$$I_1 = {}_1(a_j, \alpha_j)_n ; {}_{n+1}(a_{ji}, \alpha_{ji})_{p_i},$$

$$I_2 = {}_1(b_j, \beta_j)_m ; {}_{m+1}(b_{ji}, \beta_{ji})_{q_i},$$

$$I_3 = {}_1(c_j, \gamma_j)_N ; {}_{N+1}(c_{ji}, \gamma_{ji})_{P_i},$$

$$I_4 = (d_j, \delta_j)_M ; {}_{M+1}(d_{ji}, \delta_{ji})_{Q_i},$$

$$\begin{aligned}
 I_5 &= (1 - \mu, \gamma)_{,1} (a_j, \alpha_j)_n ,_1 (1 - d_j - \frac{\eta \delta_j}{\delta}, \frac{\sigma \delta_j}{\delta})_M ; \\
 &\quad M+1 (1 - d_{ji} - \frac{\eta \delta_{ji}}{\delta}, \frac{\sigma \delta_{ji}}{\delta})_{Q_i, n+1} (a_{ji}, \alpha_{ji})_{p_i}, (\lambda, \rho), \\
 I_6 &= (\lambda - \mu, \rho + \gamma)_{,1} (b_j, \beta_j)_m ,_1 (1 - c_j - \frac{\eta \gamma_j}{\delta}, \frac{\sigma \gamma_j}{\delta})_N ; \\
 &\quad N+1 (1 - c_{ji} - \frac{\eta \gamma_{ji}}{\delta}, \frac{\sigma \gamma_{ji}}{\delta})_{P_i, m+1} (b_{ji}, \beta_{ji})_{q_i}.
 \end{aligned}$$

The following are conditions for convergence.

- i. $\sigma > 0, \rho > 0, \delta > 0, \eta > 0$, for $i = 1, 2, \dots, r$,
- ii. $|\arg \frac{\gamma}{a}| < \pi$,
- iii. $0 < \Re(\mu) < \Re(\lambda)$,
- iv. $\Re(\lambda - \mu - \frac{(\gamma+\rho)(a_j-1)}{\alpha_j}) > 0$, for $j = 1, 2, \dots, n$,
- v. $\Re(\eta + \frac{\delta(c_i-1)}{\gamma_i} + \frac{\sigma(a_j-1)}{\alpha_j}) < 0$, for $i = 1, 2, \dots, N, j = 1, 2, \dots, n$,
- vi. $\Re(\eta + \frac{\delta(d_j)}{\delta_j} + \frac{\sigma b_i}{\beta_i}) > 0$, for $i = 1, 2, \dots, m, j = 1, 2, \dots, M$,
- vii. $\Re(\mu + \frac{\gamma b_j}{\beta_j}) > 0$, for $j = 1, 2, \dots, m$.
- viii. Let A and B are given by Eq.(6) and Eq.(7) respectively.
 - (a) $A > 0, |\arg u| < \frac{\pi A}{2}$,
 - (b) $A \geq 0, |\arg u| \geq \frac{\pi A}{2}, \Re(B + 1) < 0$.
- ix. Let A' and B' are given by Eq.(9) and Eq.(10) respectively.
 - (a) $A' > 0, |\arg b| < \frac{\pi A'}{2}$,
 - (b) $A' \geq 0, |\arg b| \geq \frac{\pi A'}{2}, \Re(B' + 1) < 0$

Proof

Let

$$\begin{aligned}
 \Delta &= \int_y^\infty \int_0^\infty (x+a)^{-\lambda} (x-y)^{\mu-1} z^{\eta-1} I_{P_i, Q_i; r}^{m, n} \\
 &\quad \left[u(x+a)^\rho (x-y)^\gamma z^\sigma \left| \begin{matrix} I_1 \\ I_2 \end{matrix} \right. \right] I_{P_i, Q_i; r}^{M, N} \left[bz^\delta \left| \begin{matrix} I_3 \\ I_4 \end{matrix} \right. \right] dx dz
 \end{aligned}$$

where I_1, I_2, I_3 and I_4 are same as in Eq.(15). Express the first I -function as a contour integral using Eq.(5) and interchange the order of integrations. Then Δ becomes:

$$\frac{1}{2\pi i} \int_L u^\xi \left\{ \int_y^\infty (x+a)^{\rho\xi-\lambda} (x-y)^{\gamma\xi+\mu-1} dx \right\} \left\{ \int_0^\infty z^{\eta+\sigma\xi-1} I_{P_i, Q_i; r}^{M, N} \left[bz^\delta \left| \begin{matrix} I_3 \\ I_4 \end{matrix} \right. \right] dz \right\} \theta(\xi) d\xi, \quad (16)$$

where $\theta(\xi)$ is same as in Eq.(5). Now being used (1.4) in (3.9) it can be reduced to:

$$\frac{b^{-\frac{\eta}{\delta}} (y+a)^{\mu-\lambda}}{\delta} \frac{1}{2\pi i} \int_L b^{-\frac{\sigma\xi}{\delta}} (y+a)^{(\gamma+\rho)\xi} u^\xi \theta(\xi) \frac{\{\Gamma(\mu+\gamma\xi)\Gamma(\lambda-\mu-(\rho+\gamma)\xi)\}}{\Gamma(\lambda-\rho\xi)} \times \frac{\prod_{j=1}^M \Gamma\left(d_j + \frac{\eta\delta_j}{\delta} + \frac{\sigma\delta_j\xi}{\delta}\right) \prod_{j=1}^N \Gamma\left(1 - c_j - \frac{\eta\gamma_j}{\delta} - \frac{\sigma\gamma_j\xi}{\delta}\right)}{\sum_{i=1}^r \left[\prod_{j=M+1}^{Q_i} \Gamma\left(1 - d_{ji} - \frac{\eta\delta_{ji}}{\delta} - \frac{\sigma\delta_{ji}\xi}{\delta}\right) \prod_{j=N+1}^{P_i} \Gamma\left(c_{ji} + \frac{\eta\gamma_{ji}}{\delta} + \frac{\sigma\gamma_{ji}\xi}{\delta}\right) \right]} d\xi. \quad (17)$$

Applying Eq.(5) in Eq.(17), the right hand side of Eq.(15) is obtained .

3.2.1 Special Cases

1. The case $r = 1$

We have

$$\int_y^\infty \int_0^\infty (x+a)^{-\lambda} (x-y)^{\mu-1} z^{\eta-1} H_{p,q}^{m,n} \left[u(x+a)^\rho (x-y)^\gamma z^\sigma \left| \begin{matrix} I_1 \\ I_2 \end{matrix} \right. \right] H_{P,Q}^{M,N} \left[bz^\delta \left| \begin{matrix} I_3 \\ I_4 \end{matrix} \right. \right] dx dz = \frac{b^{-\frac{\eta}{\delta}} (y+a)^{(\mu-\lambda)}}{\delta} H_{p+Q+2, q+P+1}^{m+N+1, n+M+1} \left[ub^{-\frac{\sigma}{\delta}} (y+a)^{(\gamma+\rho)} \left| \begin{matrix} I_5 \\ I_6 \end{matrix} \right. \right], \quad (18)$$

where

$$I_1 =_1 (a_j, \alpha_j)_p,$$

$$I_2 =_1 (b_j, \beta_j)_q,$$

$$I_3 =_1 (c_j, \gamma_j)_P,$$

$$I_4 =_1 (d_j, \delta_j)_Q,$$

$$I_5 = (1 - \mu, \gamma)_{,1} (a_j, \alpha_j)_n, {}_1 (1 - d_j - \frac{\eta\delta_j}{\delta}, \frac{\sigma\delta_j}{\delta})_Q, {}_{n+1} (a_j, \alpha_j)_p, (\lambda, \rho)$$

$$I_6 = (\lambda - \mu, \rho + \gamma)_{,1} (b_j, \beta_j)_m ,_1 (1 - c_j - \frac{\eta\gamma_j}{\delta}, \frac{\sigma\gamma_j}{\delta})_{P, m+1} (b_j, \beta_j)_q.$$

The conditions of convergence are the following:

- i. $\sigma > 0, \rho > 0, \eta > 0, \theta > 0, \theta' > 0, \epsilon > 0, \epsilon' > 0.$
- ii. $|\arg u| < \frac{1}{2}\theta\pi, |\arg b| < \frac{1}{2}\theta'\pi,$
- iii. $0 < \Re(\mu) < \Re(\lambda), |\arg \frac{y}{a}| < \pi,$
- iv. $\Re(\lambda - \mu - \frac{(\gamma+\rho)(a_j-1)}{\alpha_j}) > 0, \text{ for } j = 1, 2, \dots, n.$
- v. $\Re(\eta + \frac{\delta(c_i-1)}{\gamma_i} + \frac{\sigma(a_j-1)}{\alpha_j}) < 0, \text{ for } i = 1, 2, \dots, N, j = 1, 2, \dots, n.$
- vi. $\Re(\eta + \frac{\delta(d_j)}{\delta_j} + \frac{\sigma(b_i)}{\beta_i}) > 0, \text{ for } i = 1, 2, \dots, m, j = 1, 2, \dots, M.$

2. The case $\delta = 1$

We have

$$\int_y^\infty \int_0^\infty (x+a)^{-\lambda} (x-y)^{\mu-1} z^{\eta-1} H_{p,q}^{m,n} \left[\begin{matrix} u(x+a)^\rho (x-y)^\gamma z^\sigma \\ I_1 \\ I_2 \end{matrix} \middle| \begin{matrix} I_3 \\ I_4 \end{matrix} \right] H_{P,Q}^{M,N} \left[\begin{matrix} bz \\ I_3 \\ I_4 \end{matrix} \right] dx dz$$

$$= b^{-\eta} (y+a)^{(\mu-\lambda)} H_{p+Q+2, q+P+1}^{m+N+1, n+M+1} \left[\begin{matrix} ub^{-\sigma} (y+a)^{(\gamma+\rho)} \\ I_5 \\ I_6 \end{matrix} \right], \quad (19)$$

where

$$I_1 = {}_1(a_j, \alpha_j)_p,$$

$$I_2 = {}_1(b_j, \beta_j)_q,$$

$$I_3 = {}_1(c_j, \gamma_j)_P,$$

$$I_4 = {}_1(d_j, \delta_j)_Q$$

$$I_5 = (1 - \mu, \gamma)_{,1} (a_j, \alpha_j)_n ,_1 (1 - d_j - \eta\delta_j, \sigma\delta_j)_{Q, n+1} (a_j, \alpha_j)_p, (\lambda, \rho)$$

$$I_6 = (\lambda - \mu, \rho + \gamma)_{,1} (b_j, \beta_j)_m ,_1 (1 - c_j - \eta\gamma_j, \sigma\gamma_j)_{P, m+1} (b_j, \beta_j)_q,$$

provided the conditions are similar to that of (3.11) with $\delta = 1$. Eq.(19) agrees with the result given by Prabha Singh [4]

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