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Double integrals involving product of two I-functions

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Abstract: In this paper two double integrals involving product of two I-Functions of one variable are established by using Mellin Transform. Special cases include the results of Prabha Singh[4].

Keywords: special functions, I-functions

1 Results used

The following well known results are used in the proofs of the main results without explicit reference. Here and also elsewhere in the paper the symbol $\Re z$ shall denote the real part of the complex number z.

1. For $\Re\gamma > 0$, $\Re\mu < -\Re\gamma + 1$ we have (Erdelyi [1, p.350]),

$$\int_1^\infty x^{\mu-1}(x-1)^{\gamma-1}dx = \frac{\Gamma(\gamma)\Gamma(1-\gamma-\mu)}{\Gamma(1-\mu)}. \tag{1}$$

2. For $0 < \Re\mu < \Re\lambda$ we have (Erdelyi [2, pp.201-233]),

$$\int_y^\infty x^{-\lambda}(x-y)^{\mu-1}dx = \frac{\Gamma(\mu)\Gamma(\lambda-\mu)}{\Gamma(\lambda)}y^{\mu-\lambda}. \tag{2}$$

3. For $\Re\gamma > 0, \Re\rho > \Re\gamma$ we have,

$$\int_0^\infty x^{\gamma-1}(x+y)^{-\rho}dx = \frac{\Gamma(\gamma)\Gamma(\rho-\gamma)}{\Gamma(\rho)}y^{\gamma-\rho}. \tag{3}$$

4. For $0 < \Re\mu < \Re\lambda$, $|arg\frac{y}{a}| < \pi$ we have,

$$\int_y^\infty (x+a)^{-\lambda}(x-y)^{\mu-1}dx = \frac{\Gamma(\mu)\Gamma(\lambda-\mu)}{\Gamma(\lambda)}(y+a)^{\mu-\lambda}. \tag{4}$$

2 Introduction

The I-function of one variable is defined by Saxena [5] in the following manner:

$$I(z) = I\,{}^{m,n}_{p_i,q_i;r}\left[z\left|\begin{array}{l} {}_1(a_j,\alpha_j)_n;{}_{n+1}(a_{ji},\alpha_{ji})_{p_i} \\ {}_1(b_j,\beta_j)_m;{}_{m+1}(b_{ji},\beta_{ji})_{q_i}\end{array}\right.\right]$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right]} z^{\xi}\,d\xi, \tag{5}$$

where the parameters defining the function satisfy the following conditions:

- $p_i(i=1,2,\ldots,r)$, $q_i(i=1,2,\ldots,r)$, m, n are integers satisfying
$$0 \le n \le p_i, \quad 0 \le m \le q_i;$$
- r is finite;
- α_j, β_j, α_{ji} and β_{ji} are real and positive;
- a_j, b_j, a_{ji} and b_{ji} are complex numbers such that
$$a_j(b_h+v) \ne \beta_h(a_j-1-k),$$
for $v,k=0,1,2,\ldots$, $h=1,2,\ldots,m$, $j=1,2,\ldots,n$;
- L is a contour running from $-i\infty$ to $+i\infty$ in the complex ξ-plane such that the points
$$\xi = \frac{(a_j-1-k)}{\alpha_j}, \quad j=1,2,\ldots,n, \quad v=0,1,2,\ldots$$
$$\xi = \frac{(b_j+v)}{\beta_j}, \quad j=1,2,\ldots,m, \quad v=0,1,2,\ldots$$
lie to the left hand and right hand sides of L.

Integral (5) converges absolutely in the $\xi-$plane, if

i. $A>0$, $|\arg z| < \frac{\pi A}{2}$

ii. $A \ge 0$, $|\arg z| \ge \frac{\pi A}{2}$, $\Re(B+1)<0$

where A and B are defined as follows:

$$A = \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{m}\beta_j - \max_{1\le i\le r}\left[\sum_{j=n+1}^{p_i}\alpha_{ji} + \sum_{j=m+1}^{q_i}\beta_{ji}\right], \tag{6}$$

$$B = \sum_{j=1}^{m}b_j - \sum_{j=1}^{n}a_j - \min_{1\le i\le r}\left[\sum_{j=n+1}^{p_i}a_{ji} - \sum_{j=m+1}^{q_i}b_{ji} + \frac{p_i}{2} - \frac{q_i}{2}\right]. \tag{7}$$

3 Main results

In the next two subsections we present the main results of this paper which are two integral formulas involving the product of two I-functions. Two special cases of each of these integral formulas are also presented. One of these special cases is a result proved by Prabha Singh [4].

3.1 Integral formula 1

The following formula is one of the main results of this paper.

$$\int_y^\infty \int_0^\infty x^{-\lambda}(x-y)^{\mu-1} z^{\eta-1}\, I^{\,m,n}_{\,p_i,q_i;r} \times \left[ux^\rho (x-y)^\gamma z^{-\sigma} \,\middle|\, \begin{matrix} I_1 \\ I_2 \end{matrix} \right] I^{\,M,N}_{\,P_i,Q_i;r} \left[bz^\delta \,\middle|\, \begin{matrix} I_3 \\ I_4 \end{matrix} \right] dxdz$$
$$= \frac{b^{-\frac{\eta}{\delta}} y^{(\mu-\lambda)}}{\delta}\, I^{\,m+M+1,n+N+1}_{\,p_i+P_i+2,q_i+Q_i+1;r} \left[ub^{\frac{\sigma}{\delta}} y^{(\gamma+\rho)} \,\middle|\, \begin{matrix} I_5 \\ I_6 \end{matrix} \right], \tag{8}$$

where

$$\begin{aligned}
I_1 &= {}_1(a_j,\alpha_j)_n \,;\, {}_{n+1}(a_{ji},\alpha_{ji})_{p_i} \\
I_2 &= {}_1(b_j,\beta_j)_m \,;\, {}_{m+1}(b_{ji},\beta_{ji})_{q_i} \\
I_3 &= {}_1(c_j,\gamma_j)_N \,;\, {}_{N+1}(c_{ji},\gamma_{ji})_{P_i} \\
I_4 &= {}_1(d_j,\delta_j)_M \,;\, {}_{M+1}(d_{ji},\delta_{ji})_{Q_i} \\
I_5 &= (1-\mu,\gamma),{}_1(a_j,\alpha_j)_n ,{}_1\left(c_j+\frac{\eta\gamma_j}{\delta},\frac{\sigma\gamma_j}{\delta}\right)_N; \\
&\qquad {}_{N+1}\left(c_{ji}+\frac{\eta\gamma_{ji}}{\delta},\frac{\sigma\gamma_{ji}}{\delta}\right)_{P_i},{}_{n+1}(a_{ji},\alpha_{ji})_{p_i},(\lambda,\rho) \\
I_6 &= (\lambda-\mu,\rho+\gamma),{}_1(b_j,\beta_j)_m ,{}_1\left(d_j+\frac{\eta\delta_j}{\delta},\frac{\sigma\delta_j}{\delta}\right)_M; \\
&\qquad {}_{M+1}\left(d_{ji}+\frac{\eta\delta_{ji}}{\delta},\frac{\sigma\delta_{ji}}{\delta}\right)_{Q_i},{}_{m+1}(b_{ji},\beta_{ji})_{q_i}.
\end{aligned}$$

The following are conditions of convergence:

i. $\sigma>0,\ \rho>0,\ \eta>0,\ \gamma>0,\ \delta>0,\ \mu>0,\ \lambda>0, i=1,2,\ldots,r.$

ii. $0<\Re(\mu)<\Re(\lambda)$.

iii. $\Re\left(\lambda-\mu-\frac{(\gamma+\rho)(a_j-1)}{\alpha_i}\right)>0$ for $j=1,2,\ldots,n$.

iv. $\Re\left(\eta+\frac{\delta(c_i-1)}{\gamma_i}-\frac{\sigma b_j}{\beta_j}\right)<0$ for $i=1,2,\ldots,N,\ j=1,2,\ldots,m$.

v. $\Re\left(\eta + \frac{\delta(d_j)}{\delta_j} + \frac{\sigma(1-a_i)}{\alpha_i}\right) > 0$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, M$.

vi. $\Re\left(\mu + \frac{\gamma b_j}{\beta_j}\right) > 0$, for $j = 1, 2, \ldots, m$.

vii. Let A and B be given by Eq.(6) and Eq.(7) respectively.

(a) $A > 0$, $|\arg u| < \frac{\pi A}{2}$;
(b) $A \geq 0$, $|\arg u| \geq \frac{\pi A}{2}$, $|\Re(B+1)| < 0$.

viii. Define A' and B' by

$$A' = \sum_{j=1}^{N} \gamma_j + \sum_{j=1}^{M} \delta_j - \max_{1\leq i\leq r}\left[\sum_{j=N+1}^{P_i} \gamma_{ji} + \sum_{j=M+1}^{Q_i} \delta_{ji}\right] \tag{9}$$

$$B' = \sum_{j=1}^{M} d_j - \sum_{j=1}^{N} c_j - \min_{1\leq i\leq r}\left[\sum_{j=N+1}^{P_i} c_{ji} - \sum_{j=M+1}^{Q_i} d_{ji} + \frac{P_i}{2} - \frac{Q_i}{2}\right]. \tag{10}$$

We must have

(a) $A' > 0$, $|\arg b| < \frac{\pi A'}{2}$;
(b) $A' \geq 0$, $|\arg b| \geq \frac{\pi A'}{2}$, $\Re(B'+1) < 0$.

Proof

Let

$$\Delta = \int_y^\infty \int_0^\infty x^{-\lambda}(x-y)^{\mu-1} z^{\eta-1}\, I\,{}^{m,n}_{p_i,q_i;r}\left[ux^\rho(x-y)^\gamma z^{-\sigma}\middle|\begin{matrix} I_1 \\ I_2\end{matrix}\right] I\,{}^{M,N}_{P_i,Q_i;r}\left[bz^\delta\middle|\begin{matrix} I_3 \\ I_4\end{matrix}\right] dxdz,$$

whereI_1, I_2, I_3 and I_4 are same as in (8). Express the I-function as a contour integral using (5) and interchange the order of integrations. Then Δ becomes

$$\frac{1}{2\pi i}\int_L u^\xi \left\{\int_y^\infty x^{\rho\xi-\lambda}(x-y)^{\gamma\xi+\mu-1}dx\right\} \left\{\int_0^\infty z^{\eta-\sigma\xi-1} I\,{}^{M,N}_{P_i,Q_i;r}\left[bz^\delta\middle|\begin{matrix} I_3 \\ I_4\end{matrix}\right] dz\right\}\theta(\xi)d\xi, \tag{11}$$

where $\theta(\xi)$ is same as in Eq.(8). Now being used Eq.(2) in Eq.(11) it can be reduced to:

$$\frac{b^{-\frac{\eta}{\delta}}}{\delta} y^{\mu-\lambda}\frac{1}{2\pi i}\int_L b^{\frac{\sigma\xi}{\delta}} y^{(\gamma+\rho)\xi} u^\xi \theta(\xi)\ \frac{\{\Gamma(\mu+\gamma\xi)\Gamma(\lambda-\mu-(\rho+\gamma)\xi)\}}{\Gamma(\lambda-\rho\xi)}\times$$

$$\frac{\prod_{j=1}^{M}\Gamma\left(d_j+\frac{\eta\delta_j}{\delta}-\frac{\sigma\delta_j\xi}{\delta}\right)\prod_{j=1}^{N}\Gamma\left(1-c_j-\frac{\eta\gamma_j}{\delta}+\frac{\sigma\gamma_j\xi}{\delta}\right)}{\sum_{i=1}^{r}\left[\prod_{j=M+1}^{Q_i}\Gamma\left(1-d_{ji}-\frac{\eta\delta_{ji}}{\delta}+\frac{\sigma\delta_{ji}\xi}{\delta}\right)\prod_{j=N+1}^{P_i}\Gamma\left(c_{ji}+\frac{\eta\gamma_{ji}}{\delta}-\frac{\sigma\gamma_{ji}\xi}{\delta}\right)\right]}d\xi. \quad (12)$$

Applying (5) in (12), the right hand side of (8) is obtained.

3.1.1 Special Cases

1. **The case** $r = 1$

   Eq.(8) reduces to:

$$\int_y^\infty\int_0^\infty x^{-\lambda}(x-y)^{\mu-1}z^{\eta-1}\, H\,{}^{m,n}_{p,q}\left[ux^{\rho}(x-y)^{\gamma}z^{-\sigma}\middle|\begin{matrix}I_1\\ I_2\end{matrix}\right] H\,{}^{M,N}_{P,Q}\left[bz^{\delta}\middle|\begin{matrix}I_3\\ I_4\end{matrix}\right]dxdz$$

$$=\frac{b^{\frac{-\eta}{\delta}}y^{(\mu-\lambda)}}{\delta}\, H\,{}^{m+M+1,n+N+1}_{p+P+2,q+Q+1}\left[ub^{\frac{\sigma}{\delta}}y^{(\gamma+\rho)}\middle|\begin{matrix}I_5\\ I_6\end{matrix}\right], \quad (13)$$

   where

$$I_1 ={}_1 (a_j,\alpha_j)_p,$$
$$I_2 ={}_1 (b_j,\beta_j)_q,$$
$$I_3 = {}_1(c_j,\gamma_j)_P,$$
$$I_4 = {}_1(d_j,\delta_j)_Q\ ,$$
$$I_5 = (1-\mu,\gamma),{}_1 (a_j,\alpha_j)_n,\ {}_1\left(c_j+\frac{\eta\gamma_j}{\delta},\frac{\sigma\gamma_j}{\delta}\right)_P,\ {}_{n+1}(a_j,\alpha_j)_p,(\lambda,\rho),$$
$$I_6 = (\lambda-\mu,\rho+\gamma),{}_1 (b_j,\beta_j)_m\ ,{}_1\left(d_j+\frac{\eta\delta_j}{\delta},\frac{\sigma\delta_j}{\delta}\right)_Q,\ {}_{m+1}(b_j,\beta_j)_q.$$

   The conditions of convergence are the following.

   i. $\delta > 0, \mu > 0, \lambda > 0, \gamma > 0, \sigma > 0,\ \rho > 0, \eta > 0.$
   ii. $0 < \Re(\mu) < \Re(\lambda),$
   iii. $\Re(\lambda-\mu-\frac{(\gamma+\rho)(a_j-1)}{\alpha_i}) > 0,$ for $j = 1, 2, \ldots.$
   iv. $\Re(\eta+\frac{\delta(c_i-1)}{\gamma_i}-\frac{\sigma b_j}{\beta_j}) < 0$, for $i = 1, 2, ..., N$, $j = 1, 2, ..., m$;
   v. $\Re(\eta+\frac{\delta(d_j)}{\delta_j}+\frac{\sigma(1-a_i)}{\alpha_i}) > 0$, for $i = 1, 2, ..., n$, $j = 1, 2, ..., M$.
   vi. $\Re(\mu+\frac{\gamma b_j}{\beta_j}) > 0$, for $j = 1, 2, \ldots, m$.

2. **The case** $\delta = 1$

Eq.(13) reduces to the following.

$$\int_y^\infty \int_0^\infty x^{-\lambda}(x-y)^{\mu-1} z^{\eta-1}\, H\,{}_{p,q}^{m,n} \left[ux^\rho (x-y)^\gamma z^{-\sigma} \middle| \begin{matrix} I_1 \\ I_2 \end{matrix} \right] H\,{}_{P,Q}^{M,N} \left[bz \middle| \begin{matrix} I_3 \\ I_4 \end{matrix} \right] dxdz$$
$$= b^{-\eta} y^{(\mu-\lambda)} H_{p+P+2,q+Q+1}^{m+M+1,n+N+1} \left[ub^\sigma y^{(\gamma+\rho)} \middle| \begin{matrix} I_5 \\ I_6 \end{matrix} \right], \tag{14}$$

where

$$\begin{aligned}
&I_1 = {}_1(a_j,\alpha_j)_p,\\
&I_2 ={}_1 (b_j,\beta_j)_q,\\
&I_3 = {}_1(c_j,\gamma_j)_P,\\
&I_4 = {}_1(d_j,\delta_j)_Q\ ,\\
&I_5 = (1-\mu,\gamma),\ {}_1(a_j,\alpha_j)_n,\ {}_1(c_j+\eta\gamma_j,\sigma\gamma_j)_P,\ {}_{n+1}(a_j,\alpha_j)_p,(\lambda,\rho),\\
&I_6 = (\lambda-\mu,\rho+\gamma),{}_1(b_j,\beta_j)_m\ ,\ {}_1(d_j+\eta\delta_j,\sigma\delta_j)_Q,\ {}_{m+1}(b_j,\beta_j)_q.
\end{aligned}$$

provided the conditions are similar to that of (13) with $\delta = 1$. (14) agrees with the result given by Prabha Singh [4].

3.2 Integral Formula 2

The following formula is second main results of this paper.

$$\int_y^\infty \int_0^\infty (x+a)^{-\lambda}(x-y)^{\mu-1} z^{\eta-1}\, I\,{}_{p_i,q_i;r}^{m,n} \left[u(x+a)^\rho (x-y)^\gamma z^{\sigma} \middle| \begin{matrix} I_1 \\ I_2 \end{matrix} \right] I\,{}_{P_i,Q_i;r}^{M,N} \left[bz^\delta \middle| \begin{matrix} I_3 \\ I_4 \end{matrix} \right] dx\, dz$$
$$= \frac{b^{\frac{-\eta}{\delta}} (y+a)^{(\mu-\lambda)}}{\delta}\, I\,{}_{p_i+Q_i+2,q_i+P_i+1;r}^{m+N+1,n+M+1} \left[ub^{-\frac{\sigma}{\delta}} (y+a)^{(\gamma+\rho)} \middle| \begin{matrix} I_5 \\ I_6 \end{matrix} \right], \tag{15}$$

where

$$\begin{aligned}
&I_1 = {}_1(a_j,\alpha_j)_n\ ;\ {}_{n+1}(a_{ji},\alpha_{ji})_{p_i},\\
&I_2 ={}_1 (b_j,\beta_j)_m\ ;\ {}_{m+1}(b_{ji},\beta_{ji})_{q_i},\\
&I_3 ={}_1 (c_j,\gamma_j)_N\ ;\ {}_{N+1}(c_{ji},\gamma_{ji})_{P_i},\\
&I_4 ={}_(d_j,\delta_j)_M\ ;\ {}_{M+1}(d_{ji},\delta_{ji})_{Q_i},
\end{aligned}$$

$$I_5 = (1-\mu,\gamma)_{,1}\,(a_j,\alpha_j)_n\;_{,1}\,(1-d_j-\frac{\eta\delta_j}{\delta},\frac{\sigma\delta_j}{\delta})_M;$$
$$_{M+1}(1-d_{ji}-\frac{\eta\delta_{ji}}{\delta},\frac{\sigma\delta_{ji}}{\delta})_{Q_i},_{n+1}(a_{ji},\alpha_{ji})_{p_i},(\lambda,\rho),$$
$$I_6 = (\lambda-\mu,\rho+\gamma)_{,1}\,(b_j,\beta_j)_m\;_{,1}\,(1-c_j-\frac{\eta\gamma_j}{\delta},\frac{\sigma\gamma_j}{\delta})_N\,;$$
$$_{N+1}(1-c_{ji}-\frac{\eta\gamma_{ji}}{\delta},\frac{\sigma\gamma_{ji}}{\delta})_{P_i},_{m+1}(b_{ji},\beta_{ji})_{q_i}.$$

The following are conditions for convergence.

i. $\sigma>0\,,\ \rho>0,\ \ \delta>0,\ \ \eta>0,$ for $i=1,2,...,r,$

ii. $|\arg\ \frac{y}{a}|<\pi,$

iii. $0<\Re(\mu)<\Re(\lambda),$

iv. $\Re(\lambda-\mu-\frac{(\gamma+\rho)(a_j-1)}{\alpha_j})>0,$ for $j=1,2,\ldots,n,$

v. $\Re(\eta+\frac{\delta(c_i-1)}{\gamma_i}+\frac{\sigma(a_j-1)}{\alpha_j})<0,$ for $i=1,2,\ldots,N,\ j=1,2,\ldots,n,$

vi. $\Re(\eta+\frac{\delta(d_j)}{\delta_j}+\frac{\sigma b_i}{\beta_i})>0,$ for $i=1,2,\ldots,m,\ j=1,2,\ldots,M,$

vii. $\Re(\mu+\frac{\gamma b_j}{\beta_j})>0,$ for $j=1,2,..,m.$

viii. Let A and B are given by Eq.(6) and Eq.(7) respectively.

(a) $A>0,|\arg\ u|<\frac{\pi A}{2},$
(b) $A\geq 0,|\arg\ u|\geq\frac{\pi A}{2},\ \Re(B+1)<0.$

ix. Let A' and B' are given by Eq.(9) and Eq.(10) respectively.

(a) $A'>0,|\arg\ b|<\frac{\pi A'}{2},$
(b) $A'\geq 0,|\arg\ b|\geq\frac{\pi A'}{2},\Re(B'+1)<0$

Proof

Let

$$\Delta=\int_y^\infty\int_0^\infty (x+a)^{-\lambda}(x-y)^{\mu-1}z^{\eta-1}\,I^{\,m,n}_{\,p_i,q_i;r}\left[u(x+a)^\rho(x-y)^\gamma z^\sigma\middle|\begin{array}{c}I_1\\ I_2\end{array}\right]\,I^{\,M,N}_{\,P_i,Q_i;r}\left[bz^\delta\middle|\begin{array}{c}I_3\\ I_4\end{array}\right]dx\,dz$$

whereI_1, I_2, I_3 and I_4 are same as in Eq.(15). Express the first I-function as a contour integral using Eq.(5) and interchange the order of integrations. Then Δ becomes:

$$\frac{1}{2\pi i}\int_L u^{\xi}\left\{\int_y^{\infty}(x+a)^{\rho\xi-\lambda}(x-y)^{\gamma\xi+\mu-1}dx\right\}$$
$$\left\{\int_0^{\infty} z^{\eta+\sigma\xi-1} I\,{}^{M,N}_{P_i,Q_i;r}\left[bz^{\delta}\middle|\begin{array}{c} I_3\\ I_4\end{array}\right]dz\right\}\theta(\xi)d\xi, \qquad (16)$$

where $\theta(\xi)$ is same as in Eq.(5). Now being used (1.4) in (3.9) it can be reduced to:

$$\frac{b^{\frac{-\eta}{\delta}}}{\delta}(y+a)^{\mu-\lambda}\frac{1}{2\pi i}\int_L b^{-\frac{\sigma\xi}{\delta}}(y+a)^{(\gamma+\rho)\xi}u^{\xi}\theta(\xi)\ \frac{\{\Gamma(\mu+\gamma\xi)\Gamma(\lambda-\mu-(\rho+\gamma)\xi)\}}{\Gamma(\lambda-\rho\xi)}\times$$
$$\frac{\prod_{j=1}^{M}\Gamma\left(d_j+\frac{\eta\delta_j}{\delta}+\frac{\sigma\delta_j\xi}{\delta}\right)\prod_{j=1}^{N}\Gamma\left(1-c_j-\frac{\eta\gamma_j}{\delta}-\frac{\sigma\gamma_j\xi}{\delta}\right)}{\sum_{i=1}^{r}\left[\prod_{j=M+1}^{Q_i}\Gamma\left(1-d_{ji}-\frac{\eta\delta_{ji}}{\delta}-\frac{\sigma\delta_{ji}\xi}{\delta}\right)\prod_{j=N+1}^{P_i}\Gamma\left(c_{ji}+\frac{\eta\gamma_{ji}}{\delta}+\frac{\sigma\gamma_{ji}\xi}{\delta}\right)\right]}d\xi. \qquad (17)$$

Applying Eq.(5) in Eq.(17), the right hand side of Eq.(15) is obtained .

3.2.1 Special Cases

1. **The case** $r=1$

   We have

$$\int_y^{\infty}\int_0^{\infty}(x+a)^{-\lambda}(x-y)^{\mu-1}z^{\eta-1}H^{m,n}_{p,q}$$
$$\left[u(x+a)^{\rho}(x-y)^{\gamma}z^{\sigma}\middle|\begin{array}{c} I_1\\ I_2\end{array}\right]H^{M,N}_{P,Q}\left[bz^{\delta}\middle|\begin{array}{c} I_3\\ I_4\end{array}\right]dxdz$$
$$=\frac{b^{\frac{-\eta}{\delta}}(y+a)^{(\mu-\lambda)}}{\delta}H^{m+N+1,n+M+1}_{p+Q+2,q+P+1}\left[ub^{-\frac{\sigma}{\delta}}(y+a)^{(\gamma+\rho)}\middle|\begin{array}{c} I_5\\ I_6\end{array}\right], \qquad (18)$$

   where

$$I_1 =_1 (a_j,\alpha_j)_p,$$
$$I_2 =_1 (b_j,\beta_j)_q,$$
$$I_3 =_1 (c_j,\gamma_j)_P,$$
$$I_4 =_1 (d_j,\delta_j)_Q,$$
$$I_5 = (1-\mu,\gamma),_1(a_j,\alpha_j)_n\ ,_1(1-d_j-\frac{\eta\delta_j}{\delta},\frac{\sigma\delta_j}{\delta})_Q,\ _{n+1}(a_j,\alpha_j)_p,(\lambda,\rho)$$

$$I_6 = (\lambda-\mu, \rho+\gamma),_1 (b_j, \beta_j)_m \;,_1 (1-c_j-\frac{\eta\gamma_j}{\delta}, \frac{\sigma\gamma_j}{\delta})_P \;,_{m+1} (b_j, \beta_j)_q.$$

The conditions of convergence are the following:

i. $\sigma > 0,\ \rho > 0, \eta > 0, \theta > 0, \theta' > 0, \epsilon > 0, \epsilon' > 0.$

ii. $|\arg u| < \frac{1}{2}\theta\pi, |\arg b| < \frac{1}{2}\theta'\pi,$

iii. $0 < \Re(\mu) < \Re(\lambda), |\arg \frac{y}{a}| < \pi,$

iv. $\Re(\lambda-\mu-\frac{(\gamma+\rho)(a_j-1)}{\alpha_j}) > 0,$ for $j = 1, 2, \ldots, n.$

v. $\Re(\eta+\frac{\delta(c_i-1)}{\gamma_i}+\frac{\sigma(a_j-1)}{\alpha_j}) < 0,$ for $i = 1, 2, \ldots, N, j = 1, 2, \ldots, n.$

vi. $\Re(\eta+\frac{\delta(d_j)}{\delta_j}+\frac{\sigma(b_i)}{\beta_i}) > 0,$ for $i = 1, 2, \ldots, m, j = 1, 2, \ldots, M.$

2. **The case** $\delta = 1$

We have

$$\int_y^\infty \int_0^\infty (x+a)^{-\lambda}(x-y)^{\mu-1} z^{\eta-1} H_{p,q}^{m,n} \left[u(x+a)^\rho (x-y)^\gamma z^\sigma \middle| \begin{matrix} I_1 \\ I_2 \end{matrix} \right] H_{P,Q}^{M,N} \left[bz \middle| \begin{matrix} I_3 \\ I_4 \end{matrix} \right] dxdz$$
$$= b^{-\eta}(y+a)^{(\mu-\lambda)} H_{p+Q+2,q+P+1}^{m+N+1,n+M+1} \left[ub^{-\sigma}(y+a)^{(\gamma+\rho)} \middle| \begin{matrix} I_5 \\ I_6 \end{matrix} \right], \qquad (19)$$

where

$I_1 = {}_1(a_j, \alpha_j)_p,$
$I_2 = {}_1(b_j, \beta_j)_q,$
$I_3 =_1 (c_j, \gamma_j)_P,$
$I_4 = {}_1(d_j, \delta_j)_Q$
$I_5 = (1-\mu, \gamma),_1 (a_j, \alpha_j)_n \;,_1 (1-d_j-\eta\delta_j, \sigma\delta_j)_Q, \;_{n+1}(a_j, \alpha_j)_p, (\lambda, \rho)$
$I_6 = (\lambda-\mu, \rho+\gamma),_1 (b_j, \beta_j)_m \;,_1 (1-c_j-\eta\gamma_j, \sigma\gamma_j)_P \;,_{m+1} (b_j, \beta_j)_q,$

provided the conditions are similar to that of (3.11) with $\delta = 1$. Eq.(19) agrees with the result given by Prabha Singh [4]

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